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# Rigorous and Fast Constraints Transformations at the Solution Level: Case Studies for Regional and Global GNSS Networks

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## Abstract

The study introduces an efficient methodology to perform the transformations between station coordinate and velocity solutions where either minimum or redundant datum constraints have been imposed employing the estimated state vector and the covariance matrix thereof. The analytical methodology presented herein facilitates the datum alignment of large-network solutions, especially for the GNSS technique. The computational complexity reduction is achieved by avoiding the expensive normal equation system reconstruction and the subsequent inversion thereof, which is the current norm, in favor of an elegant approach involving the inversion of an up to 14-order matrix. All information parsed in our algorithm is readily available in the widely used space geodetic solution files following the Solution Independent Exchange (SINEX) format. Our transformation approach is evaluated in two globally distributed GNSS-derived solutions and one terrestrial reference frame with a spatial concentration in South America. The results prove the equivalence of the current and proposed algorithm and that our approach is at least an order of magnitude faster. In addition, we test the Fast Constraints Transformation (FCT) through simulated networks, with a size of up to

5000 stations. The FCT presented here accelerates the transformation by almost 140 times compared to the commonly used strategy.

**Keywords**: least squares adjustment, constraints, terrestrial reference frames, GNSS network, SINEX, Fast Constraint Transformation

## Introduction

The International Reference Frame (ITRF) is a set of reference point trajectories on the earth's surface that realize a terrestrial reference system (Bosy 2014, Crespi et al. 2015). The reference system definition and its realization play a crucial role for the modern Terrestrial Reference Frame (TRF) accuracy (Altamimi and Dermanis 2009, Collilieux et al. 2013). There are numerous studies addressing the reference system definition and its realization (Davies and Blewitt 2000, Sillard and Boucher 2001). One of the most important issues for the TRF realization is the role of the constraints that provide information related to the definition of the origin, orientation and scale of the system. By and large, the datum constraints fall into either of two categories:

The so-called Minimum Constraints (MC) are imposed only to treat the problem of the rank deficiency (Baarda 1973) of the Normal Equations (NEQ) related to a Least-Squares adjustment problem (Kotsakis 2012). In essence, they provide information that is not provided inherently by the measurements involved to the adjustment. We may also point out that the terms MC and inner constraints (Pope 1971, Leick et al. 2015) are interchangeable for the TRF solution strategy.

The Over Constraints (OC), also known as redundant constraints, correspond to a number of constraint equations larger than the rank deficiency (Dermanis 1987). They are usually considered to add extra constraints to an existing geodetic network. OC could be beneficial for various cases, e.g., when a daily regional network should be rigourously expressed in an ITRF (Sánchez et al. 2011).

In general, the OC solution imposes a particular geometry on the existing geodetic network. It is possible that one may want to transform an OC to an MC solution in order to study and compare the influence of different estimation strategies or to obtain an MC solution, e.g., for a combined multi-technique global Terrestrial Reference Frame TRF. We should clarify that in the context of this study, the use of constraints refers to the so-called No Net Conditions (NNCs), namely: No Net Translation (NNT), No Net Rotation (NNR) and No Net Scale (NNS) (Angermann et al. 2004). The NNCs are mainly used for the global terrestrial reference frame construction (Altamimi et al. 2002). These conditions are directly expressed through similarity transformation (Helmert) parameters; usually, orientation and origin and/or scale and/or orientation, which define the origin (NNT), define the orientation (NNR) and define the scale (NNS). Within the geodetic literature, there are many studies referring to the TRFs constraint handling (Davies and Blewitt 2000).

The classical approach builds up a rigorous transformation between two solutions; the exact steps are described in the next section with a comparison with the fast transformation. However, there is a serious drawback pertaining to the computational effort and complexity in the case of large networks. The previously described rigorous approach is based on the inversion of potentially large NEQ systems, especially for the GNSS TRF solution case

In the present study, we describe a new method, the Fast Constraints Transformation (FCT), that allows the rigorous transformation between MC and OC solutions and vice versa as well as among different MCs. Taking advantage of the plethora of previous studies, we collect and analyze all the crucial points regarding the easier implementation of handling constraints through the prism of the modern TRFs. In essence, the FCT proposed herein presents an efficient way of not interfering with the Normal Equations (NEQ), thus saving computational effort. We prove that one can directly work at the solution level for many practical applications if they adopt the proposed methodology. The main advantage of FCT lies in the inversion of a relatively small matrix, not exceeding the size of 14. We give all the necessary mathematical formulas for any user who is willing to follow the algorithm as well as ready-to-deploy software. In addition, we show the possibility to apply the method in the so-called solution level derived from any given SINEX file.

The FCT is validated through a weekly solution of the South American TRF called SIRGAS (Sanchez et al. 2010). Furthermore, we test FCT methodology to a. two global GNSS networks of the International GNSS Service (IGS) and b. global simulated GNSS networks. The results of FCT methodology are compared with those of the classical approach, confirming the success of the methodology.

## Methodology for the Transformation of the Least-Squares Solutions for TRFs

We begin with the well-known formula of the free NEQ system (Koch 1987):

$$\mathbf{N}\mathbf{x} = \mathbf{u} \tag{1}$$

where $\mathbf{x} = \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{o}}$ is the vector of the corrections to the a priori $\mathbf{x}^{\mathbf{o}}$ of the unknown parameters $\mathbf{x}^{\mathbf{a}}$, $\mathbf{N} = \mathbf{A}^{\mathbf{T}}\mathbf{Q}\mathbf{A}$ denotes the NEQ singular matrix with the design matrix or Jacobian $\mathbf{A}$ not being of full rank for any space geodetic technique if station coordinate estimates are sought, $\mathbf{u} = \mathbf{A}^{\mathbf{T}}\mathbf{Q}\mathbf{b}$ denotes the right-hand side vector, and $\mathbf{Q}$ is a proper weight matrix. To handle the rank-deficiency, one can impose MC:

$$(\mathbf{N} + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{B}}\mathbf{B})\mathbf{x} = \mathbf{u} \tag{2a}$$

or in a compact form

$$\mathbf{N}^{\mathbf{MC}}\mathbf{x} = \mathbf{u} \tag{2b}$$

In the case of the OC solution, one can write:

$$(\mathbf{N} + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{B}}\mathbf{B} + \mathbf{G}^{\mathrm{T}}\mathbf{Q}_{\mathbf{G}}\mathbf{G})\mathbf{x} = \mathbf{u} \tag{3a}$$

or

$$(\mathbf{N}^{\mathbf{MC}} + \mathbf{G}^{\mathrm{T}}\mathbf{Q}_{\mathbf{G}}\mathbf{G})\mathbf{x} = \mathbf{u} \tag{3b}$$

or

$$\mathbf{N}^{\mathbf{OC}}\mathbf{x} = \mathbf{u} \tag{3c}$$

where $\mathbf{N}^{\mathbf{MC}}$ is the non-singular NEQ matrix of the MC solution, and $\mathbf{N}^{\mathbf{OC}}$ is the non-singular NEQ matrix of the OC solution. $\mathbf{B} = (\mathbf{E}^{\mathrm{T}}\mathbf{E})^{-\mathbf{1}}\mathbf{E}^{\mathrm{T}}$ is the matrix which relieves the initial NEQ matrix of its inherent rank deficiency (Davies and Blewitt 2000, Wu et al. 2015) and it is also used for the ITRF construction (Altamimi et al. 2002). The Helmert matrix $\mathbf{E}$ contains all the appropriate columns for the MC solution. The rows refer to $m$ stations. Matrix $\mathbf{E}$ has the general form (Leick and van Gelder 1975, Altamimi et al. 2002):

$$\mathbf{E} = \begin{bmatrix} \mathbf{T}_{\mathbf{i}} & \mathrm{D}_{\mathbf{i}} & \mathbf{R}_{\mathbf{i}} \\ \vdots & \vdots & \vdots \\ \mathbf{T}_{\mathbf{m}} & \mathrm{D}_{\mathbf{m}} & \mathbf{R}_{\mathbf{m}} \end{bmatrix} \tag{4}$$

where

$$\mathbf{T_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5)$$

refers to the origin parameters $[\mathrm{T_X} \quad \mathrm{T_Y} \quad \mathrm{T_Z}]^{\mathbf{T}}$,

$$\mathbf{D_i} = [\mathrm{x_i} \quad \mathrm{y_i} \quad \mathrm{z_i}]^{\mathbf{T}} \quad (6)$$

refers to the scale parameter **D**, and,

$$\mathbf{R_i} = \begin{bmatrix} 0 & \mathrm{z_i} & -\mathrm{y_i} \\ -\mathrm{z_i} & 0 & \mathrm{x_i} \\ \mathrm{y_i} & -\mathrm{x_i} & 0 \end{bmatrix} \quad (7)$$

refers to the orientation parameters. The rows of **E** matrix for the stations which are not marked as fiducial hence not included in the subset of MC definition are null. The triplet $[\mathrm{x_i} \quad \mathrm{y_i} \quad \mathrm{z_i}]^{\mathbf{T}}$ refers to the 3D coordinates of an arbitrary network node $i$. The size of the **E** matrix is $m \times k$, where $m$ is the total number of the unknown coordinates and/or velocities parameters and $k$ the number of the Helmert parameters which refer to specific No Net Conditions (NNC) as $\mathbf{B}^{\mathrm{T}}\mathbf{Q_B}\mathbf{B} = \mathbf{0}$. The term NNC refers to individual No Net Conditions: NNT, NNR, and NNS and/or to their combination. The maximum number of the **E** matrix columns are 14, expressing the NNC for the coordinates and velocities.

**G** in (3) refers to a matrix that attributes the associated columns that correspond to particular Helmert parameters of the **E** matrix, which are used for the OC solution. The size of **B** and **G** matrices are $m \times k_1$ and $m \times k_2$, respectively. The term $k_1$ refers to the number of the Helmert parameters of the MC, while $k_2$ to the number of the Helmert parameters constrained for the OC. Finally, $\mathbf{Q_B}$ and $\mathbf{Q_G}$ are the square and usually diagonal weight matrices of the Helmert parameters for MC and OC solutions, respectively. The sizes of these two matrices are $k_1$ X $k_1$ for the MC and $k_2$ X $k_2$ for the OC case, respectively. In the case of selecting a set of stations instead all of them, the **E** matrix will have zero rows for the stations not involved to the MC datum definition The use of a stations sub-set, instead of all for the MC definition, is called partial inner constraints (Kotsakis 2013). We should underline that for the present study, the terms MC and OC refer to weighted NNC, which are mainly applied to the TRF studies.

As a final comment, we should point out that the solution of the MC at (2a) corresponds to zero constraints ($\mathbf{B}^{\mathrm{T}}\mathbf{Q_B}\mathbf{B}$) which are commonly used in the global reference frame realization. However, one can apply non-zero constraints such as changing the Euler pole for the new plate reference frame; this method is currently not applicable as the non-zero constraints are not considered within the current study.

Transformation from Minimum Constraints to Over Constraints

We proceed with the description of the transformation from MC to OC following some specific mathematical relations. The derivation of the equations is based on the analytical relations of the constraints used for this purpose.

*Parameter Estimation for OC*

Following (3), the solution of the OC is written as:

$$\hat{\mathbf{x}}^{\mathbf{OC}} = \mathbf{C}^{\mathbf{OC}}\mathbf{u} \quad (8)$$

where

$$\mathbf{C}^{\mathbf{OC}} = (\mathbf{N}^{\mathbf{MC}} + \mathbf{G}^{\mathrm{T}}\mathbf{Q}_{\mathbf{G}}\mathbf{G})^{-\mathbf{1}} \quad (9)$$

denotes the covariance matrix (CV) of the OC solution.

Expanding (9) according to the matrix identities, we have now (Blewitt 1998, Chapter 6):

$$\underbrace{\mathbf{C}^{\mathbf{OC}}}_{\mathrm{m\,x\,m}} = \underbrace{\mathbf{C}^{\mathbf{MC}}}_{\mathrm{m\,x\,m}} - \underbrace{\mathbf{C}^{\mathbf{MC}}}_{\mathrm{m\,x\,m}} \underbrace{\mathbf{G}^{\mathrm{T}}}_{\mathrm{m\,x}\,k_2} \left( \underbrace{\mathbf{C}_{\mathbf{G}}}_{k_2\,\mathrm{x}\,k_2} + \underbrace{\mathbf{G}}_{k_2\,\mathrm{x\,m}} \underbrace{\mathbf{C}^{\mathbf{MC}}}_{\mathrm{m\,x\,m}} \underbrace{\mathbf{G}^{\mathrm{T}}}_{\mathrm{m\,x}\,k_2} \right)^{-\mathbf{1}} \underbrace{\mathbf{G}}_{k_2\,\mathrm{x\,m}} \underbrace{\mathbf{C}^{\mathbf{MC}}}_{\mathrm{m\,x\,m}} \quad (10)$$

where $\mathbf{C}^{\mathbf{MC}} = (\mathbf{N}^{\mathbf{MC}})^{-\mathbf{1}} = (\mathbf{N} + \mathbf{B}^{\mathbf{T}}\mathbf{Q}_{\mathbf{B}}\mathbf{B})^{-\mathbf{1}}$ and $\mathbf{C}_{\mathbf{G}} = (\mathbf{Q}_{\mathbf{G}})^{-\mathbf{1}}$**;** the bottom line presents the matrix dimension. Combining (9) and (10) yields:

$$\hat{\mathbf{x}}^{\mathbf{OC}} = \mathbf{C}^{\mathbf{MC}}\mathbf{u} - \mathbf{C}^{\mathbf{MC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C}_{\mathbf{G}} + \mathbf{G}\mathbf{C}^{\mathbf{MC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G}\mathbf{C}^{\mathbf{MC}}\mathbf{u} \quad (11)$$

Considering that $\hat{\mathbf{x}}^{\mathbf{MC}} = \mathbf{C}^{\mathbf{MC}}\mathbf{u}$**,** (11) yields:

$$\hat{\mathbf{x}}^{\mathbf{OC}} = (\mathbf{I} - \mathbf{C}^{\mathbf{MC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C}_{\mathbf{G}} + \mathbf{G}\mathbf{C}^{\mathbf{MC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G})\hat{\mathbf{x}}^{\mathbf{MC}} \Rightarrow$$

$$\hat{\mathbf{x}}^{\mathbf{OC}} = \mathbf{L}\hat{\mathbf{x}}^{\mathbf{MC}} \quad (12)$$

Using (12) the CV matrix could be also expressed as follows by applying the error propagation law:

$$\mathbf{C}^{\mathbf{OC}} = \mathbf{L}\mathbf{C}^{\mathbf{MC}}\mathbf{L}^{\mathbf{T}} \quad (13)$$

According to (12) we successfully transformed an MC solution to an OC solution, using the estimated vector of the unknowns, its associated CV matrix, and matrices $\mathbf{G}$ and $\mathbf{Q_G}$. The vector of the estimated unknowns and its associated CV matrix have already been computed and probably stored, e.g., in a SINEX file. The new term is the matrix $\mathbf{C_G} + \mathbf{G}\mathbf{Q_G}\mathbf{G}^{\mathrm{T}}$ which should be inverted. In other words, one can save lot of computational effort by inverting a small matrix instead of a matrix of a much larger dimension. For example, the CV matrix size of the GNSS network of ITRF2014 (Altamimi et al. 2016) is $17616 \times 17616$. The IGS combination scheme includes nowadays large number of stations (Rebischung et al. 2016). Equation (8) is applied with respect to a unique MC solution. A different set of MC will lead to different solutions. So, the described algorithm transforms a unique MC to OC solution(s).

The OC solution should be carefully implemented since it is directly related to the geometry of the geodetic networks (Dermanis 1987). For example, one can constrain the scale of a GNSS and a DORIS solution to test the origin and scale definitions of these two techniques with respect to an ITRF solution. Through FCT this could be easily realized, avoiding large matrix inversions. For the cases of regional GNSS networks, FCT comprises advantages, as we will discuss next.

The MC to OC transformation presented herein refers exclusively for rank deficient systems such as VLBI with respect to the origin and the orientation (rank deficiency of 6). Similarly, there is no need to include a Z-rotation parameter for GNSS solutions without orbit parameters.

*Estimating the a posteriori variance factor for the OC solution*

We begin with the optimization criterion $\widehat{\varphi}_{\mathbf{MC}}$ for the MC, which is the minimization of the weighted squared residuals (Koch,1987):

$$\widehat{\varphi}_{\mathbf{MC}} = \mathbf{b}^{\mathbf{T}}\mathbf{Q}\mathbf{b} - \mathbf{u}^{\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{MC}} \quad (14)$$

where $\mathbf{b}$ the vector of the reduced observations also known as the observed-minus-computed vector, and $\mathbf{Q}$ is the weight of the observations. The definition for the a posteriori variance factor is:

$$\widehat{\sigma}^2_{\mathrm{MC}} = \frac{\widehat{\varphi}_{\mathbf{MC}}}{\mathrm{f}_{\mathrm{MC}}} = \frac{\hat{\mathbf{e}}^{\mathbf{T}}\mathbf{P}\hat{\mathbf{e}}}{\mathrm{f}_{\mathrm{MC}}} \quad (15)$$

where $\hat{\sigma}^2_{\mathrm{MC}}$ is the a-posteriori variance factor, $\hat{\mathbf{e}}$ is the vector of the residuals and $\mathrm{f}_{\mathrm{MC}}$ denotes the degrees of freedom. From (14) we get:

$$\mathbf{b}^{\mathbf{T}}\mathbf{P}\mathbf{b} = \hat{\varphi}_{\mathbf{MC}} + \mathbf{u}^{\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{MC}} \tag{16}$$

Recasting (14) employing (12) and (16) for the OC case, we have:

$$\hat{\varphi}_{\mathbf{OC}} = \hat{\varphi}_{\mathbf{MC}} + \mathbf{u}^{\mathrm{T}}(\mathbf{I}-\mathbf{L})\hat{\mathbf{x}}_{\mathrm{MC}} \tag{17}$$

Equivalently:

$$\hat{\varphi}_{\mathbf{OC}} = \hat{\sigma}^2_{\mathrm{MC}}\mathrm{f}_{\mathrm{MC}} + \mathbf{u}^{\mathrm{T}}(\mathbf{I}-\mathbf{L})\hat{\mathbf{x}}_{\mathbf{MC}} \tag{18}$$

If we substitute $\mathbf{u} = \mathbf{C}^{\mathbf{MC}}\hat{\mathbf{x}}_{\mathbf{MC}}$ **,** equation (18) now reads:

$$\hat{\varphi}_{\mathbf{OC}} = \hat{\sigma}^2_{\mathrm{MC}}\mathrm{f}_{\mathrm{MC}} + (\hat{\mathbf{x}}_{\mathrm{MC}})^{\mathbf{T}}\mathbf{C}^{\mathbf{MC}}(\mathbf{I}-\mathbf{L})\hat{\mathbf{x}}_{\mathbf{MC}} \tag{19}$$

Finally, the a-posteriori variance for the OC solution will be estimated as follows:

$$\hat{\sigma}^2_{\mathrm{OC}} = \frac{\hat{\varphi}_{\mathbf{MC}}+\mathbf{u}^{\mathrm{T}}(\mathbf{I}-\mathbf{L})\hat{\mathbf{x}}_{\mathrm{MC}}}{\mathrm{f}_{\mathrm{MC}}+\mathrm{k}} = \frac{\hat{\sigma}^2_{\mathrm{MC}}\mathrm{f}_{\mathrm{MC}}+\mathbf{u}^{\mathrm{T}}(\mathbf{I}-\mathbf{L})\hat{\mathbf{x}}_{\mathrm{MC}}}{\mathrm{f}_{\mathrm{MC}}+\mathrm{k}} = \frac{\hat{\sigma}^2_{\mathrm{MC}}\mathrm{f}_{\mathrm{MC}}+(\hat{\mathbf{x}}_{\mathrm{MC}})^{\mathbf{T}}\mathbf{C}^{\mathbf{MC}}(\mathbf{I}-\mathbf{L})\hat{\mathbf{x}}_{\mathrm{MC}}}{\mathrm{f}_{\mathrm{MC}}+\mathrm{k}} \tag{20}$$

where $k$ is the number of the OC. Taking into account the given quantities of a SINEX file, it is possible to achieve the transition from the MC to OC using the described.

Transformation from OC to MC

Let us re-write (3a) as $\mathbf{N}^{\mathbf{OC}} = \mathbf{N} + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_{\mathbf{B}}\mathbf{B} + \mathbf{G}^{\mathrm{T}}\mathbf{Q}_{\mathbf{G}}\mathbf{G}$ , from which it follows that

$$\mathbf{N}^{\mathbf{MC}} = \mathbf{N}^{\mathbf{OC}} - \mathbf{G}^{\mathrm{T}}\mathbf{Q}_{\mathbf{G}}\mathbf{G} \tag{21}$$

The associated covariance matrix of the estimated unknowns is:

$$\mathbf{C}^{\mathbf{MC}} = (\mathbf{N}^{\mathbf{OC}} - \mathbf{G}^{\mathrm{T}}\mathbf{Q}_{\mathbf{G}}\mathbf{G})^{-\mathbf{1}} \tag{22}$$

Using the matrix identities, equation (22) yields:

$$\underbrace{\mathbf{C}^{\mathbf{MC}}}_{\mathrm{m\,x\,m}} = \underbrace{\mathbf{C}^{\mathbf{OC}}}_{\mathrm{m\,x\,m}} + \underbrace{\mathbf{C}^{\mathbf{OC}}}_{\mathrm{m\,x\,m}}\underbrace{\mathbf{G}^{\mathrm{T}}}_{\mathrm{m\,x}\,k_2}\left(\underbrace{\mathbf{C}_{\mathbf{G}}}_{k_2\,\mathrm{x}\,k_2} - \underbrace{\mathbf{G}}_{k_2\,\mathrm{x\,m}}\underbrace{\mathbf{C}^{\mathbf{OC}}}_{\mathrm{m\,x\,m}}\underbrace{\mathbf{G}^{\mathrm{T}}}_{\mathrm{m\,x}\,k_2}\right)^{-\mathbf{1}}\underbrace{\mathbf{G}}_{k_2\,\mathrm{x\,m}}\underbrace{\mathbf{C}^{\mathbf{OC}}}_{\mathrm{m\,x\,m}} \tag{23}$$

The estimated vector of the unknowns with respect to the OC solution is:

$$\hat{\mathbf{x}}^{\mathbf{MC}} = \mathbf{C}^{\mathbf{OC}}\mathbf{u}+\mathbf{C}^{\mathbf{OC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C}_{\mathbf{G}} - \mathbf{G}\mathbf{C}^{\mathbf{OC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G}\mathbf{C}^{\mathbf{OC}}\mathbf{u} \tag{24}$$

Taking into account that $\hat{\mathbf{x}}^{\mathbf{OC}} = \mathbf{C}^{\mathbf{OC}}\mathbf{u}$**,** equation (24) yields:

$$\hat{\mathbf{x}}^{\mathbf{MC}} = (\mathbf{I} + \mathbf{C}^{\mathbf{OC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C}_{\mathbf{G}} - \mathbf{G}\mathbf{C}^{\mathbf{OC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G})\hat{\mathbf{x}}^{\mathbf{OC}} \text{ or}$$

$$\hat{\mathbf{x}}^{\mathbf{MC}} = \mathbf{M}\hat{\mathbf{x}}^{\mathbf{OC}} \quad (25)$$

where $\mathbf{M} = \mathbf{I} + \mathbf{C}^{\mathbf{OC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C}_{\mathbf{G}} - \mathbf{G}\mathbf{C}^{\mathbf{OC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G}$**.** Then the CV matrix of the unknowns is:

$$\mathbf{C}^{\mathbf{MC}} = \mathbf{M}\mathbf{C}^{\mathbf{OC}}\mathbf{M}^{\mathbf{T}} \quad (26)$$

We may also note that (23) and (25) transform a unique OC solution to a unique MC solution. The relation between OC and MC is unique, considering the definition of a specific MC. The described procedure could be realized using a loosely constrained solution instead of OC. We note that loose constraints are a special group of OC whose uncertainty is relatively large.

Finally, we should also refer to the case of transformation among different MCs, utilizing the so-called S-transformation (Baarda 1973). This kind of transformation is already extensively discussed in the geodetic literature (Blaha 1971, Teunissen 1985). The S-transformation is an effective and fast tool for transforming different MC solutions without inverting large NEQ systems.

*Estimating the a posteriori variance factor for the MC solution*

Following the rationale presented before, we have the following relations:

$$\hat{\varphi}_{\mathbf{MC}} = \hat{\varphi}_{\mathbf{OC}} + \mathbf{u}^{\mathrm{T}}\hat{\mathbf{x}}_{\mathrm{OC}} - \mathbf{u}^{\mathrm{T}}\mathbf{M}\hat{\mathbf{x}}_{\mathrm{OC}} \quad \text{, giving}$$

$$\hat{\varphi}_{\mathbf{MC}} = \hat{\varphi}_{\mathbf{OC}} + \mathbf{u}^{\mathrm{T}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathbf{OC}} = \hat{\varphi}_{\mathbf{OC}} + (\hat{\mathbf{x}}_{\mathrm{OC}})^{\mathbf{T}}\mathbf{C}^{\mathbf{OC}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathrm{OC}} \quad (27)$$

and

$$\hat{\sigma}^2_{\mathrm{MC}} = \frac{\hat{\varphi}_{\mathbf{OC}} + \mathbf{u}^{\mathrm{T}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathbf{OC}}}{\mathrm{f}_{\mathrm{OC}} - \mathrm{k}} = \frac{\hat{\sigma}^2_{\mathrm{OC}}\mathrm{f}_{\mathrm{OC}} + \mathbf{u}^{\mathrm{T}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathbf{OC}}}{\mathrm{f}_{\mathrm{OC}} - \mathrm{k}} = \frac{\hat{\sigma}^2_{\mathrm{OC}}\mathrm{f}_{\mathrm{OC}} + (\hat{\mathbf{x}}_{\mathbf{OC}})^{\mathbf{T}}\mathbf{C}^{\mathbf{OC}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathbf{OC}}}{\mathrm{f}_{\mathrm{OC}} - \mathrm{k}} \quad (28)$$

A summarized description of the transformation between the different constraints handling is given in Table 1.

**Table 1** Estimated quantities after applying FCT methodology.

| **Transition** | **MC-->OC** | **OC-->MC** |
|---|---|---|
| number of over-constraints | $k_2$ | $k_2$ |

| | | |
|---|---|---|
| degrees of freedom | $\mathrm{f_{MC}}$ | $\mathrm{f_{OC}}$ |
| over-constraint matrix | $\mathbf{G}$ | $\mathbf{G}$ |
| auxiliary matrix | $\mathbf{L} = (\mathbf{I} - \mathbf{C^{MC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C_G} + \mathbf{G}\mathbf{C^{MC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G})$ | $\mathbf{M} = (\mathbf{I} + \mathbf{C^{OC}}\mathbf{G}^{\mathrm{T}}(\mathbf{C_G} - \mathbf{G}\mathbf{C^{OC}}\mathbf{G}^{\mathrm{T}})^{-\mathbf{1}}\mathbf{G})$ |
| CV matrix of the unknowns | $\mathbf{C^{OC}} = \mathbf{L}\mathbf{C^{MC}}\mathbf{L^T}$ | $\mathbf{C^{MC}} = \mathbf{M}\mathbf{C^{OC}}\mathbf{M^T}$ |
| estimated parameters | $\hat{\mathbf{x}}^{\mathbf{OC}} = \mathbf{L}\hat{\mathbf{x}}^{\mathbf{MC}}$ | $\hat{\mathbf{x}}^{\mathbf{MC}} = \mathbf{M}\hat{\mathbf{x}}^{\mathbf{OC}}$ |
| optimization criterion | $\hat{\sigma}^2_{\mathrm{MC}}\mathrm{f_{MC}} + (\hat{\mathbf{x}}_{\mathbf{MC}})^{\mathbf{T}}\mathbf{C^{MC}}(\mathbf{I} - \mathbf{L})\hat{\mathbf{x}}_{\mathbf{MC}}$ | $\hat{\sigma}^2_{\mathrm{OC}}\mathrm{f_{OC}} + (\hat{\mathbf{x}}_{\mathbf{OC}})^{\mathbf{T}}\mathbf{C^{OC}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathbf{OC}}$ |
| a posteriori variance | $\dfrac{\hat{\sigma}^2_{\mathrm{MC}}\mathrm{f_{MC}} + (\hat{\mathbf{x}}_{\mathbf{MC}})^{\mathbf{T}}\mathbf{C^{MC}}(\mathbf{I} - \mathbf{L})\hat{\mathbf{x}}_{\mathbf{MC}}}{\mathrm{f_{MC}} + \mathrm{k}}$ | $\dfrac{\hat{\sigma}^2_{\mathrm{OC}}\mathrm{f_{OC}} + (\hat{\mathbf{x}}_{\mathbf{OC}})^{\mathbf{T}}\mathbf{C^{OC}}(\mathbf{I} - \mathbf{M})\hat{\mathbf{x}}_{\mathbf{OC}}}{\mathrm{f_{OC}} - \mathrm{k}}$ |

Regional TRFs

The regional TRFs can be considered densifications of a global TRF (Bruyninx et al. 2001, Soler and Marshall 2003). They are realized mainly by GNSS observations. However, in the case of regional TRF there is usually no rank deficiency since the orbits of the GNSS satellites are fixed (Davies and Blewitt 2000) since the estimation of orbits is unreliable given the limited extent of the network. Nevertheless, the alignment to the ITRF is necessary since the initial solution of the GNSS networks is not reliable enough (Dach et al. 2015), due to inherited scale and origin definition problems. Thus, the final solution of a regional network should be tailored to the ITRF by (a) using direct Helmert transformation (Blewitt et al. 2013), (b) constraining the regional solution to an ITRF (Altamimi 2003, Sanchez 2010), or (c) using the MC approach (Altamimi 2003) as implemented from Legrand and Bruyninx (2009) and Kenyeres et al. (2019).

The FCT could also be applied to the regional GNSS TRF. The final alignment to ITRF could be done using (8)–(20). The latter treatment could be beneficial for many ACs, since they will only apply for each daily GNSS solution an inversion of a matrix of maximum size $7 \times 7$ (from equations 10 or 23), instead of a much more computationally complex inversion. The same holds for the regional TRF realizations, which include many stations in some cases such as the USA, central Europe, China. For example, suppose one wants to impose a different set of constraints to align the daily/weekly/final solutions to a regional or regional TRF. In that case, there is no need to invert a relatively large matrix. On the other hand, one can apply the FCT for removing the extra constraints, leading to the original solution derived from the initial GNSS analysis. This procedure resembles the OC to MC transformation, as previously discussed.

The alternative approach, FCT, could stand as a beneficial solution for regional ACs or national agencies in the case of limited computational resources. Thus, a regional or regional daily or weekly densification could be easily achieved, e.g., of an ITRF or a regional TRF.

Classical approach versus FCT

Thus far, if one needs to apply another type or set of constraints, one should use the NEQs exclusively to provide a solution. The most common output of a SINEX file is the CV matrix of the estimated parameters. For changing the constraints, the following well-known algorithm, the classical approach, should be followed:

1. Inversion of the CV matrix to compute the constrained solution
2. Removal of the constraints
3. Construction of the Helmert matrix of the OC/MC
4. Imposing new types of constraints
5. Solution of the new NEQ system

FCT requires the construction of a Helmert matrix of the OC and then the direct implementation of the relations described above. So, the procedure reads as follows:

1. Construction of the Helmert matrix of the OC/MC
2. Inversion of a relatively small matrix and multiplications

Table 2 summarizes the characteristics between the classical approach and the FCT for the cases of the transformation between MC and OC and vice versa and among different MC, respectively. In the later section, we present the computation load differences due to the complexity differences between these two classical and the FCT approaches using simulated and real SINEX products.

**Table 2** Characteristics of the classical approach and the FCT implementation for the transformation between MC to OC and vice versa, also for the regional network cases. The following symbols are used with respect to the computational effort: **!**: significant computational effort, ● : less computational effort

| Case | Classical approach | FCT | Computational steps: classical approach | Computational steps: FCT |
|---|---|---|---|---|
| Transformation from an MC to one or more OC | ! | ● | 5 | 2 |
| Transformation from the OC to the original MC | ! | ● | 5 | 2 |
| Transformation from the loosely constrained solution to the original MC | ! | ● | 5 | 2 |
| Handling constraint solutions for Regional GNSS networks | ! | ● | 5 | 2 |
| A posteriori variance factor | ! | ● | 5 | 2 |

Regarding the FCT we should underline the following feature. When a SINEX file contains an OC solution, in most cases, it refers to coordinates and velocities constraints for selected sites of the network rather than explicit Helmert transformation parameters, in contrast to the concept of this study. A specific block inside the SINEX file indicates the constrained sites. This fact ought to be taken into account since the FCT cannot be applied when the constraints are imposed, as previously mentioned (constrained sites). Thus, the SINEX used for the FCT must contain explicit Helmert-type constraints as a priori information.

## Numerical Tests

In this section, we deal with the implementation of the FCT described previously. We will exploit all the necessary mathematical formulas analyzed before to transform the constraints.

### The SIRGAS network

We test the FCT of constraints transformation through an arbitrarily selected weekly solution of the regional TRF of South America, the SIRGAS (Sanchez et al. 2010). We choose the weekly solution including the day of year (DOY) 134-140, 2012 (ftp://ftp.sirgas.org/pub/gps/SIRGAS/1688/, SINEX file; SIR16887.SNX) The weekly solution of SIRGAS SINEX file is estimated with loose constraints. Figure 1 depicts the SIRGAS stations included in the weekly solution.

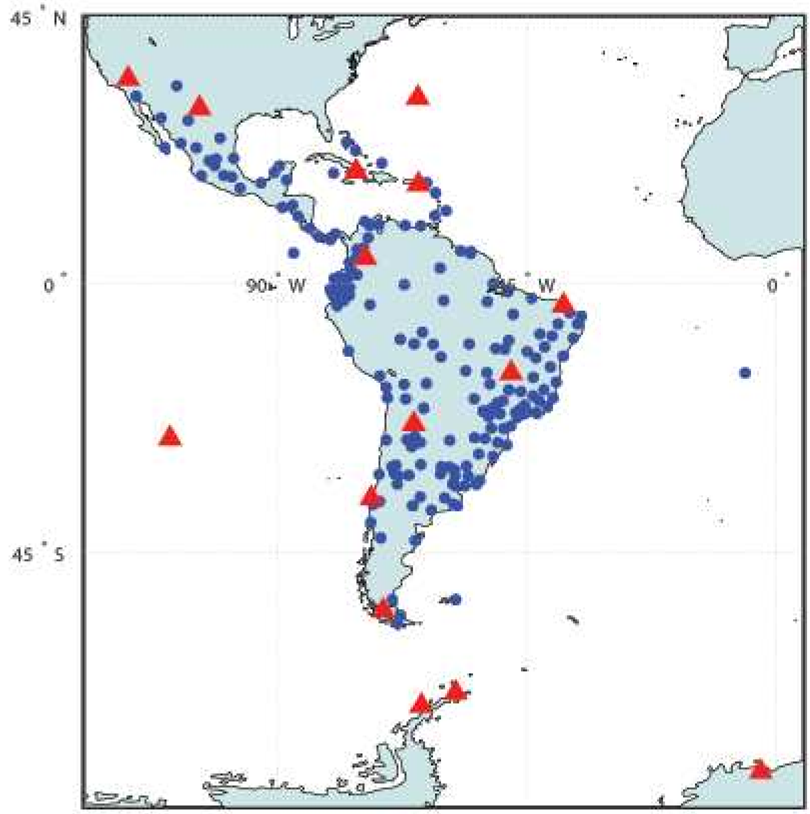


**Fig. 1** SIRGAS network consists of 203 stations. The red triangles are 15 fiducial stations; their coordinates participate in the no-net conditions. Blue dots are non-fiducial sites.

*From the initial network to OC*

We first invert the CV matrix to recover the NEQ matrix. Then we remove the loose constraints (Seitz et al. 2012, Rebischung et al. 2016), which are explicitly given in the SINEX file. After removing the loose constraints, we apply the NNT and NNR conditions and solve the NEQ systems, as the classical approach dictates (Rossikopoulos 1999). On the other hand, we proceed with the FCT using the CV after removing the loose constraints and the estimation vector x.

*From OC to MC*

For this kind of test, we expand the NEQ system of the initial solution after the removal of the loose constraints in terms of three translations (Kotsakis and Chatzinikos 2017).

This expansion leads to a rank deficiency of 3. Then, we solve the singular NEQ system imposing NNT and NNR conditions, resulting in an OC solution.

Similar to the previous case study, we remove the NNR conditions from the OC NEQ for the classical approach solution. For the FCT, we use the estimation vector and the CV matrix obtained directly from the OC, transformed to MC.

Figure 3 confirms that the FCT proposed herein provides identical results to the classical approach. The presented deviations between these two approaches are at the numerical truncation error level and can be ignored.

Global IGS network of 1053 stations

The second test is realized through a daily GNSS network (IGS repro3) for GPS week 1930, day 0, obtained from https://cddis.nasa.gov/archive/gnss/products/repro3/1930/.. The combined daily SINEX refers to an MC solution. We compare the classical and the FCT, respectively, by implementing NNT and NNR. Both OCs weights are 0.1 mm. Figure 2 depicts the IGS network, while Figure 3 summarizes and visualizes the differences between the classical approach and the FCT applied in SIRGAS and IGS Repro3 networks, respectively. The results are practically identical since the differences between the classical approach and the FCT are at the level of $10^{-13}$ mm.

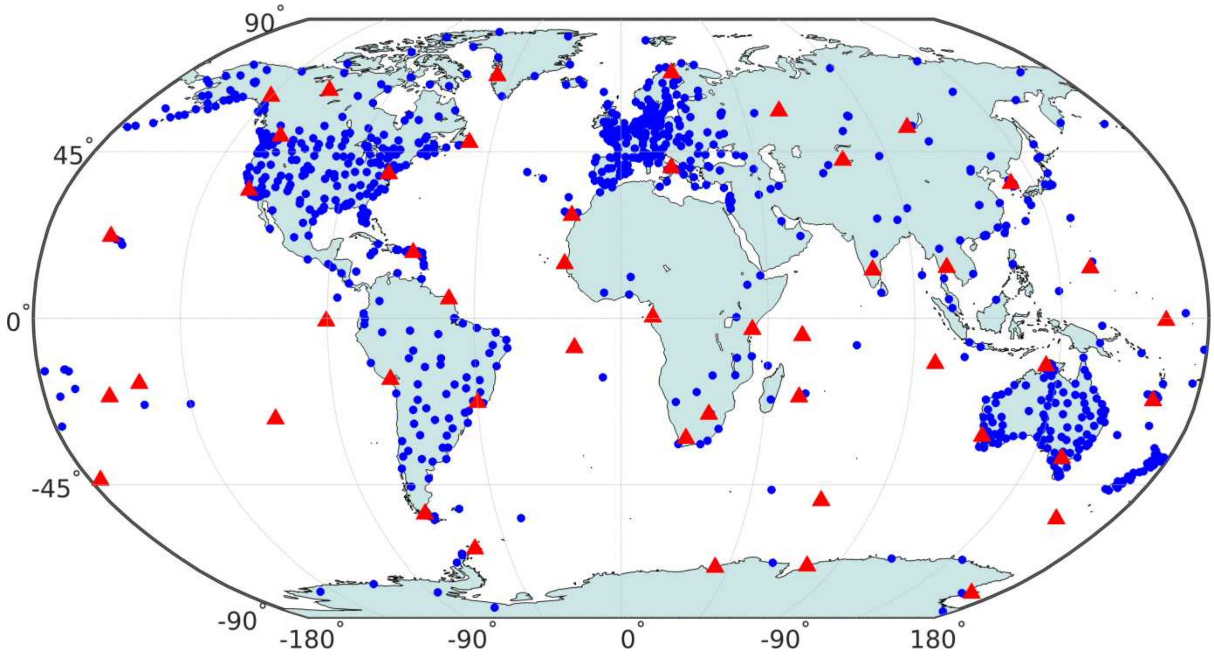

**Fig 2** IGS REPRO3 network consists of 1053 stations. The red triangles are 48 fiducial stations for NNT and NNR conditions, and blue dots are the non-fiducial stations

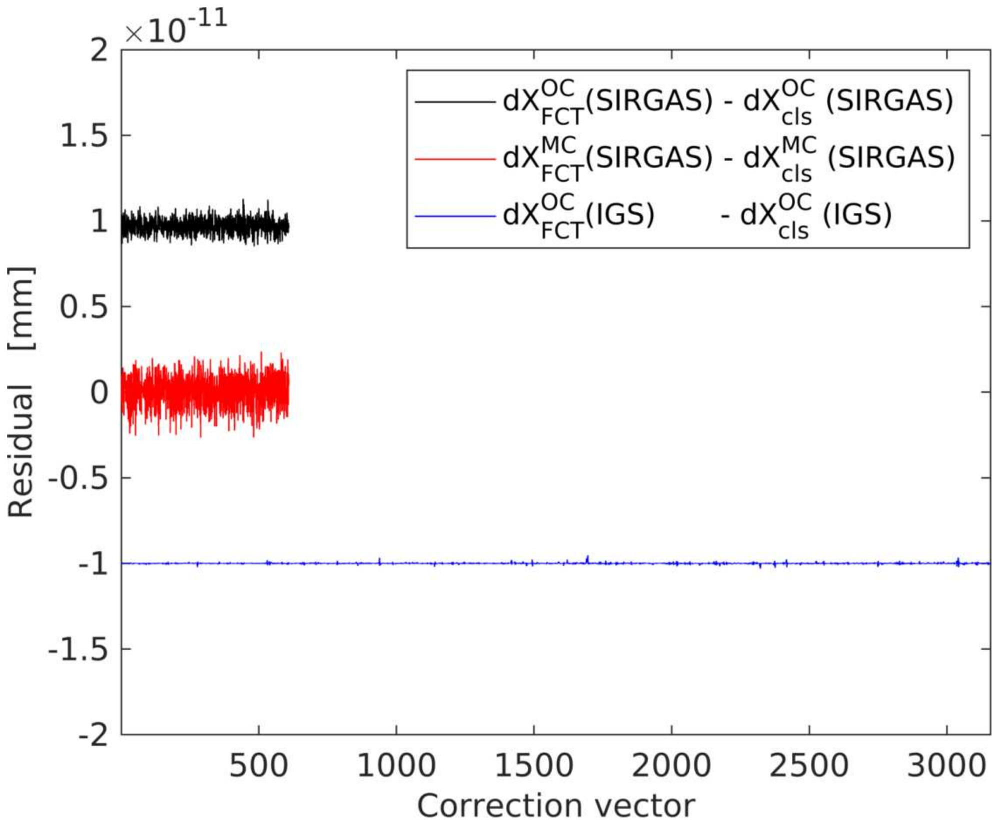


**Fig. 3:** Differences of the coordinates between the estimations from the classical approach and from Fast Constrain Transformation (FCT). The values are in the unit of mm and shifted with $+1 \times 10^{-11}$ mm for validation of MC to OC in SIRGAS network, the residual in IGS REPRO3 network is shifted with $-1 \times 10^{-11}$ mm for better visualization. The correction vector refers to the **x** vector is described in equation 1.

Comparison of Computational Complexity

We designed a simulation study to demonstrate the computation effort in the classical approach and FCT with a CV matrix, $\mathbf{C} \sim \mathcal{N}(\mu = 0, \sigma^2 = 4.10^{-6}) + 10^{-7}\mathbf{I}$, which is consisted of a random symmetric matrix filled with normal distribution. The purpose of the identity matrix is to avoid singularity because the full rank is not guaranteed in the generation of the random value. The correction term follows $\mathbf{x} \sim \mathcal{N}(0, 0.02^2)$. The fiducial sites are chosen arbitral with every third station. Then we repeat the validation of MC to OC presented above, but here we compare the computation time taken from classical and FCT for network sizes from 100 to 5 000 stations.

The simulation study is designed in MATLAB2020b. All matrix inversions call the “inv” function. Since the simulation is focusing on the ratio between two approaches, as long as the inversion methods are consistent between the comparisons, addressing the efficiency of the inversion script would be beyond the scope of this study. The simulation is performed in RedHat 8 installed in a WMware environment equipped with an Intel(R) Xeon(R) Gold 6140M CPU. This CPU is running with sequential computation on a single CPU core in comparison simulation to avoid influences from parallelization.

The number of stations in the network increases by 100 up to 1 000; then the increasing step is 1000. The values presented in Figure 4 are obtained from averaging computation time from ten repeated computations. Besides the simulations, the measured computation time, obtained from the SINEX data, is also visualized in Figure 4, including the SIRGAS and IGS network, previously described. We also included another set of computations based on measuring the computation time of these two approaches with another IGS network (igs20P21251_all.snx), including 636 stations, from https://cddis.nasa.gov/archive/gnss/products/2125/. As the algorithm is already validated with a small and a large network, we do not present the validation with this solution, but the computation effort for transferring constraints is worth demonstrating here.

We choose the simulation of up to 5 000 stations to give a conclusive discussion within a reasonable computation time. Considering the increasing number of stations, and there will definitely be even more in the future, the discussion on the computation effort is necessary and practical for large networks.

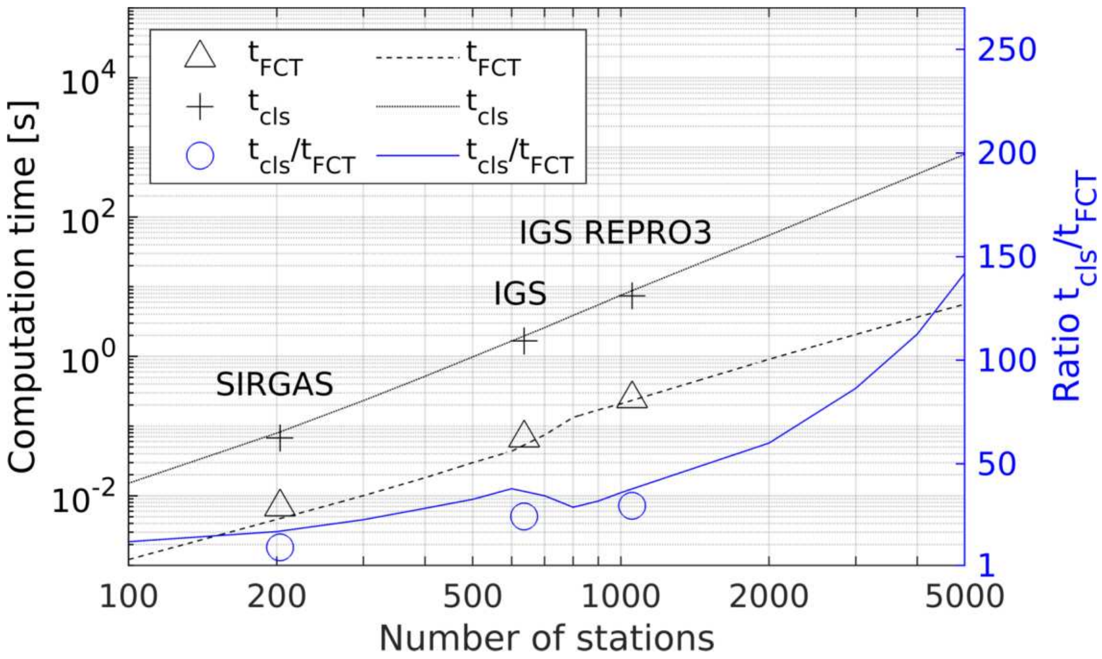


**Fig. 4** Computation times for transforming the constraints in the classical and FCT method in logarithmic scale (total time elapsed for the estimation of the Least Squares solution vector **x** for the classical approach and FCT, respectively). The markers are computation time taken from the SINEX files, with their network labels written on top of those markers. The right axis presents the ratio between the computation times from two approaches, with a linear scale in blue color.

We can see important clues regarding the computational efficiency of the FCT method. First, the computation effort of the alternative approach increases slowly with respect to the number of stations. On the other hand, the computational time increases significantly for the classical approach. For the case of the IGS Repro3 network, which consists of 1053 stations, the alternative approach needs only about 1/30 of the computation time. In a large network, the alternative approach saves significant computational effort. For the simulated network consisting of 5000 stations, the ratio between classical and alternative approaches, respectively, exceeds 140.

Figure 4 gives the sense of the trend between the computational ratios of the approaches. The fast approach shows a crucial advantage of significantly reduced computation time for a global network of thousands of stations.

## Conclusions

This study demonstrates a new FCT method for an easier and less computationally complex transformation between MC and OC solutions for global or regional TRFs. The FCT has the crucial advantage of the inversion of a relatively small matrix (maximum size of $14 \times 14$) instead of the inversion of large NEQ systems that involve matrices whose order is the thousands. The FCT is applicable to the solution level given the estimated parameters and their CV matrix and is tailored to exploit the information already available in SINEX files. We give all the necessary mathematical formulations for the transition of the estimated quantities for both cases, i.e., MC to OC and OC to MC.

The results of the FCT are compared with those of the classical approach (de-constraining and NEQ re-constructing). We have tested the FCT to a) SIRGAS (regional) network and b) large global IGS networks. It is proven that the two approaches yield identical results. In addition, we found significant improvement in the FCT computation efficiency, which can be significantly decreased (about 1/30) for the cases of a large global network, including thousands of stations. According to our simulation, FCT needs only 1/140 of the computation time when applied to the network of 5000 stations.

FCT method could serve as a beneficial procedure to many TRF-related applications, including but not limited to:

- Applying an OC solution to the case of GNSS
- Transforming an OC solution to MC
- Re-computation of a specified MC-solution from OC or loosely-constrained solution.
- For global networks with a large number of stations the FCT significantly reduces the computation effort.
- For the cases of regional TRFs the methodology shows significant advantages since the final product can be considered an OC solution (defining a stable TRF).
- Again, FCT could be applied for regional networks removing the imposed constraints and deriving the initial GNSS network (TRF fixed to the orbits).

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## Data availability

a. SIRGAS: ftp://ftp.sirgas.org/pub/gps/SIRGAS/
b. IGS repro3: https://cddis.nasa.gov/archive/gnss/products/repro3/

operational IGS combined solution: https://cddis.nasa.gov/archive/gnss/products/

The simulation dataset and corresponding software are published at the GitHub page: https://github.com/drlwang/falcons.

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